Strings in general backgrounds and the low energy effective action

We saw that the massless modes of the bosonic string theory contains $G_{\mu\nu}$, $B_{\mu\nu}$, ϕ

graviton

Since the graviton is the fluctuation of the space time geome- we can consider a string theory action in a general background $g_{\mu\nu}$

$$S = \frac{1}{2\pi\alpha'}\int d^2\sigma \sqrt{h}\, h^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu\, \cancel{\eta_{\mu\nu}}\, g_{\mu\nu}(X)$$

If one writes down the most general action for $X^\mu(\sigma,\tau)$ that is invariant under reparametrization of the string world sheet and renormalizable by power counting

with up to two derivatives of X

Tachyon ~ super renormalizable
massive states ~ non renormalizable
generate all terms corresponding to all massive states

$$S_1 = \frac{1}{2\pi\alpha'}\int d^2\sigma \sqrt{h}\, h^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu\, g_{\mu\nu}(X^\rho)$$

$$S_2 = \frac{1}{2\pi\alpha'}\int d^2\sigma\, \epsilon^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu\, B_{\mu\nu}(X^\rho)$$

$S_1 + S_2$ describes a string theory in a background with $g_{\mu\nu}$ and $B_{\mu\nu}$.

S_2 should be compared with the coupling of the charged particle to the electromagnetic field $\int A_\tau d\tau$

$$\int A_\tau d\tau = \int \frac{dX^\mu}{d\tau} A_\mu \cdot d\tau$$

Thus we see that the string is charged under $B_{\mu\nu}$.

It's rather subtle to describe a coupling of the dilaton, but it turns out to be

two deriv of h

$$S_3 = \frac{1}{4\pi}\int d^2\sigma \sqrt{h}\, \Phi(X)\, R^{(2)}$$

dimension count

world-sheet curvature

Note that $\frac{1}{4\pi}\int d^2\sigma\sqrt{h}\,R^{(2)}$ is a topological invariant

$$= \chi = 2(1-g)$$

g: genus of a Riemann surface

In the quantum effective action, this induces a term

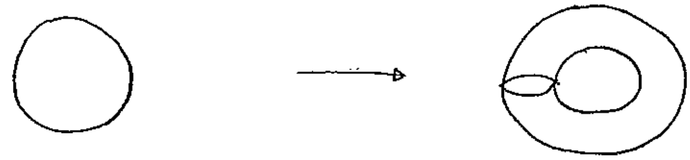
$$e^{-S_3} \sim e^{-\phi\, 2(1-g)} = e^{\phi(-2+2g)}$$ for a constant ϕ

Thus e^{ϕ} is the coupling constant for the string perturbation theory.

tree

one loop

(this kind of thickning of the field theory diagram.
we have seen in the calculation of the vacuum amplitude



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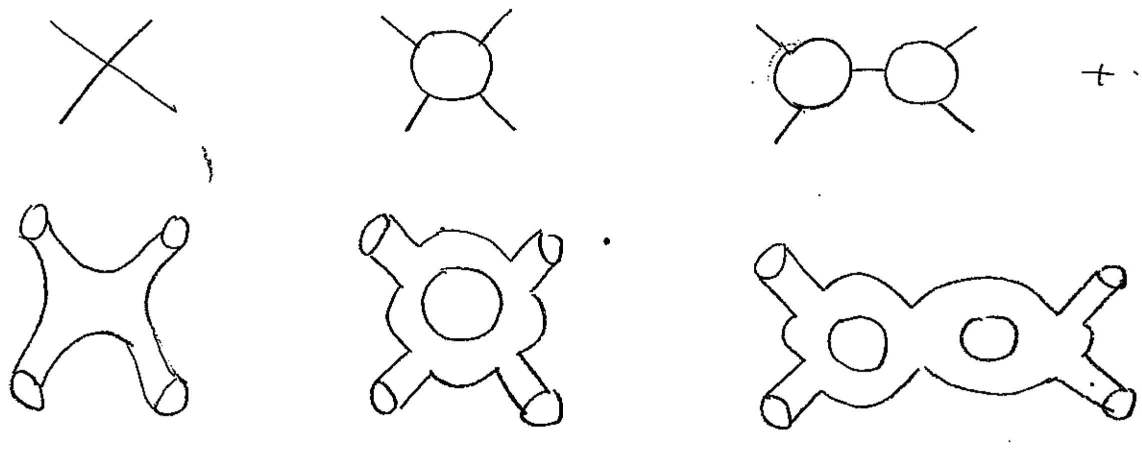
One can see that string diagrams involve all the Riemann surfaces and if we have many handles, the corresponding diagrams describe higher loops.



One can derive the equation of motion for $g_{\mu\nu}$, $B_{\mu\nu}$ and Φ.
The basic idea is that the string theory has the

classically the action → <u>conformal invariance</u> and from $S = S_1 + S_2 + S_3$

(← action at page 1 reparametrization invariances + Weyl scaling)

this also holds for the actions in pg

One can calculate the beta function for the space time dependent coupling $g_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ and $\Phi(X)$.

The result is at 1 loop

$$\beta^{g}_{\mu\nu} = R_{\mu\nu} + \frac{1}{4} H_{\mu}{}^{\lambda\rho} H_{\nu\lambda\rho} - 2 D_\mu D_\nu \Phi = 0$$

$$D_\lambda H^\lambda{}_{\mu\nu} - 2 (D_\lambda \Phi) H^\lambda{}_{\mu\nu} = 0$$

$$4 (D_\mu \Phi)^2 - 4 D_\mu D^\mu \Phi + R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} = 0$$

where $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu}$

and D_μ is covariant derivative in D-dimension

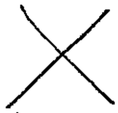
This can be derived from the 26-D action

$$S_{26} = -\frac{1}{2k^2} \int d^{26}x \sqrt{-g}\, e^{-2\Phi} (R + 4 D_\mu \Phi D^\mu \Phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho})$$

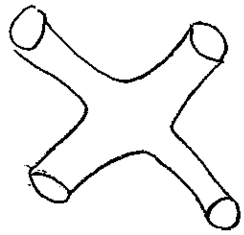
(convention: $- + + +$ $\quad R^\alpha{}_{\beta\gamma\delta} = \frac{\partial \Gamma^\alpha{}_{\beta\delta}}{\partial x^\gamma} - \frac{\partial \Gamma^\alpha{}_{\beta\gamma}}{\partial x^\delta} + \cdots \quad G_{\mu\nu} = 8\pi G T_{\mu\nu}$.

Recipe for the string amplitude

When we calculate the scattering amplitude, it involves a Feynman diagram like

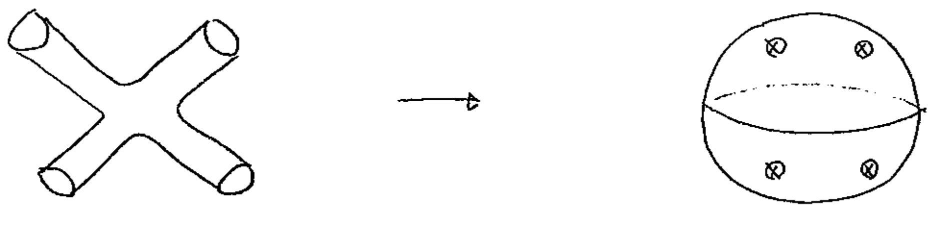


Corresponding string diagram is



It looks complicated to calculate such diagram.

What saves us is (again) conformal invariance so that we can carry out the conformal transformation.
We can shrink each leg at the above diagram to a point



In this figure, a scattering state is represented as a local ~~operator~~ insertion at a point. This operator is called a vertex operator.

A vertex operator taking momentum k has the form

$$\text{Oscillator part} \otimes e^{ik\cdot X}$$

dependent on the corresponding state

e.g. tachyon $:e^{ik\cdot X}:$

graviton $:\zeta_{\mu\nu}\partial_\alpha X^\mu \partial^\alpha X^\nu e^{ik\cdot X}:$ $\quad k^\mu \zeta_{\mu\nu}=0$

antisymmetric field $: b_{\mu\nu}\epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu e^{ik\cdot X}:$

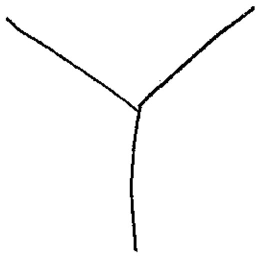
($\zeta_{\mu\nu}$ $b_{\mu\nu}$ suitable polarization te

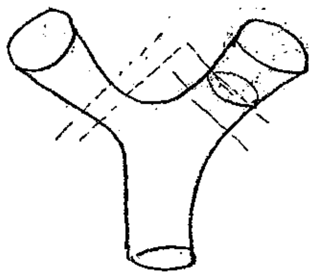
where the momentum k should satisfy the on-shell condition

Actual string amplitude involves the evaluation

$$\int \mathcal{D}X(\sigma)\,\mathcal{D}h_{\alpha\beta}\, e^{-(S_1+S_2+S_3)} \int d^2\sigma\sqrt{h}\, e^{ik\cdot X}\int d^2\sigma\sqrt{h}\,\partial_\alpha X^\mu\partial^\alpha X^\nu e^{ik\cdot X}$$

$\cdots$





When one describes the interactions, there's a difference between field theory and the string theory.

When a point particle splits into two, there is a well-defined Lorentz invariant notion of the space-time point at which the splitting occurred. It is simply the interaction vertex in the Feynman diagram. However, when a string splits into two, there is no well-defined notion of when and where this happened. In particular on any time slice for the splitting of the string, it looks like a free string propagation. While in the field theory we should define the rule for the vertex in addition to the particle propagation, once we define the free string theory, it also determines the interactions of the string theory! There are as many interacting string theories as free string theories.